# q-Serre relations in $U_q(u_n)$ and q-deformed meson mass sum rules



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LETTER TO THE EDITOR

# $q$-Serre relations in $U_q(u_n)$ and $q$-deformed meson mass sum rules

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**Abstract.** The $q$-Serre relations are shown to be necessary for fixing uniquely the value of deformation parameters within recently proposed applications of quantum algebras $U_q(u_n)$ in obtaining $q$-analogues of hadron mass sum rules. Coefficients in these $q$-analogues are expressed through Alexander polynomials of certain knots.

Use of the quantum algebra $su_q(2)$ in describing spectra of heavy nuclei is based on such requisites as the Casimir operator and Clebsch–Gordan coefficients [1–2]. In attempting to find applications of higher-rank quantum algebras, one encounters new features absent in $su_q(2)$ case. One such feature is the necessity to deal with non-simple-root elements of those algebras or, equivalently, with $q$-Serre relations. A recently proposed application of the $q$-algebras $U_q(u_n)$ to obtain $q$-analogues of hadron mass relations [3] uses both simple-root elements and non-simple-root elements. The goal of the present letter is to clarify the concrete role played by the $q$-Serre relations for that specific application.

There exist different approaches to $SU(n)$ symmetry breaking necessary for obtaining mass sum rules (MSRs) for hadrons with $n$ quark flavours. The approach based on dynamical unitary groups allows one to obtain [4–5] the following series of MSRs for vector mesons $1^-$ $(2 < n \leqslant 6)$:

$$\frac{k(k-1)}{2} m_{\omega_{k^2-1}} + \frac{k(k-1)-4}{2} m_\rho = (k-1)^2 m_{D_k^*} + \sum_{i=3}^{k-1} m_{D_i^*} \qquad k = 3, \ldots, n \tag{1}$$

where $D_k^*$ denote the isodoublets $D_3^* \equiv K^*$, $D_4^* \equiv D^*$, $D_5^* \equiv D_b^*$ and $D_6^* \equiv D_t^*$. If $n = 3$, this series begins with the famous Gell-Mann–Okubo (GMO) mass relation [6] $3m_{\omega_8} + m_\rho = 4m_{K^*}$. A comparison of this octet MSR with the existing data requires mixing between the isosinglet $\omega_8$ and the $SU(3)$ singlet, that is, $\omega_8$ is considered as a superposition of $\phi$ and $\omega$ with some mixing angle determined from the fit. Likewise, in cases of more flavours, $n > 3$, one needs $n-2$ mixing angles.

Extending that approach to quantum algebras $U_q(su_n)$, one can derive q-deformed MSRs which contain equations (1) as the $q = 1$ limit, but which also admit (if $|q| = 1$, $q \neq \pm 1$) an alternative treatment [3] without manifest singlet mixing. Let us consider some facts.

Quantum (universal enveloping) algebras $U_q(gl_n)$ are generated by the elements $\mathbf{1}$, $A_{jj}$, $A_{jj+1}$, $A_{j+1j}$, $j = 1, 2, \ldots, n-1$, which satisfy the relations [7]

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$$\begin{aligned}
&[A_{ii}, A_{jj}] = 0 \\
&[A_{ii}, A_{jj+1}] = \delta_{ij} A_{ij+1} - \delta_{ij+1} A_{ji} \\
&[A_{ii}, A_{j+1j}] = \delta_{ij+1} A_{ij} - \delta_{ij} A_{j+1i} \\
&[A_{ii+1}, A_{j+1j}] = \delta_{ij}[A_{ii} - A_{i+1i+1}]_q \\
&[A_{ii+1}, A_{jj+1}] = [A_{i+1i}, A_{j+1j}] = 0 \qquad \text{for} \quad |i-j| \geqslant 2
\end{aligned} \tag{3}$$

and the $q$-Serre relations

$$(A_{i\mp1i})^2 A_{ii\pm1} - [2]_q A_{i\mp1i} A_{ii\pm1} A_{i\mp1i} + A_{ii\pm1}(A_{i\mp1i})^2 = 0 \tag{4a}$$

$$(A_{ii\pm1})^2 A_{i\mp1i} - [2]_q A_{ii\pm1} A_{i\mp1i} A_{ii\pm1} + A_{i\mp1i}(A_{ii\pm1})^2 = 0 \tag{4b}$$

where we denoted $[B]_q \equiv (q^B - q^{-B})/(q - q^{-1})$. The 'compact' quantum algebra $U_q(u_n)$ is singled out by means of the *-operation

$$(A_{jj})^* = A_{jj} \qquad (A_{j+1j})^* = A_{jj+1} \qquad (A_{jj+1})^* = A_{j+1j}. \tag{5}$$

As for the non-deformed algebra $u_n$, finite-dimensional representations of $U_q(u_n)$ are given by sets of ordered integers $\boldsymbol{m}_n = (m_{1n}, m_{2n}, ..., m_{nn})$ and realized by means of the ($q$-analogue of) Gel'fand–Tsetlin basis and formulas. Representation formulas for $A_{ii}$ remain unchanged, and $A_{kk+1}$, $A_{k+1k}$, $k = 1, ..., n-1$, act according to formulas given in [7]. To possess action formulas for the operators which correspond to non-simple-root elements, we turn to $q$-Serre relations (4). Defining

$$A_{kk+2} = A_{kk+2}(q) \equiv q^{1/2} A_{kk+1} A_{k+1k+2} - q^{-1/2} A_{k+1k+2} A_{kk+1} \tag{6a}$$

$$A_{k+2,k} = A_{k+2,k}(q) \equiv q^{1/2} A_{k+1k} A_{k+2,k+1} - q^{-1/2} A_{k+2,k+1} A_{k+1k} \tag{6b}$$

we find that the corresponding $q$-Serre relations (e.g. with upper signs in (4 *a*), (4 *b*)) follow from the commutation rules (CR)

$$q^{1/2} A_{k+1k+2} A_{kk+2} - q^{-1/2} A_{kk+2} A_{k+1k+2} = 0 \tag{7a}$$

$$q^{1/2} A_{kk+2} A_{kk+1} - q^{-1/2} A_{kk+1} A_{kk+2} = 0. \tag{7b}$$

Dual definitions $\tilde{A}_{kk+2} \equiv -A_{kk+2}(q^{-1})$, $\tilde{A}_{k+2,k} \equiv -A_{k+2,k}(q^{-1})$ are paired with the respective dual CRs. Operators for other non-simple-root elements are treated similarly.

A mass operator commuting with the 'isospin' $U_q(su_2)$ for $3 \leqslant n \leqslant 6$ has the form [3]

$$\begin{aligned}
\hat{M}_n = M_o^{(n)} &+ \gamma_n A_{nn+1} A_{n+1n} + \delta_n A_{n+1n} A_{nn+1} \\
&+ \sum_{i=3}^{n-1} (\gamma_i A_{in+1} \tilde{A}_{n+1i} + \delta_i \tilde{A}_{n+1i} A_{in+1} + \tilde{\gamma}_i \tilde{A}_{in+1} A_{n+1i} + \tilde{\delta}_i A_{n+1i} \tilde{A}_{in+1}).
\end{aligned} \tag{8}$$

It is Hermitean, term by term, if $q$ is real. For $q = e^{ih}$, $h \in \boldsymbol{R}$, Hermiticity of mass operator requires that $\gamma_i = \tilde{\gamma}_i$, $\delta_i = \tilde{\delta}_i$. The latter choice is preferable for us.

With (8), using (Gelfand–Tsetlin basis) states for mesons from $(n^2 - 1)$-plet of 'flavour' $U_q(u_n)$ embedded into an $\{(n+1)^2 - 1\}$-plet of 'dynamical' $U_q(u_{n+1})$, one obtains

$$\begin{aligned}
&m_\rho = M_0 \qquad m_{K^*} = M_0 - \gamma_3 \qquad m_{\bar{K}^*} = M_0 - \delta_3 \\
&m_{\omega_8} = M_0 - \frac{[2]_q}{[3]_q}(\gamma_3 + \delta_3) \\
&m_{D^*} = M_0 + \gamma_4 \quad m_{\bar{D}^*} = M_0 + \delta_4 \quad m_{F^*} = M_0 - \delta_3 + \gamma_4 \quad m_{\bar{F}^*} = M_0 - \gamma_3 + \delta_4 \\
&m_{\omega_{15}} = M_0 + \left([2]_q - \frac{[3]_q}{[4]_q} - \frac{[4]_q}{[3]_q}\right)(\gamma_3 + \delta_3) + \frac{[3]_q}{[4]_q}(\gamma_4 + \delta_4)
\end{aligned} \tag{9}$$

in the four-flavour case and analogous expressions for $n = 5$ and $n = 6$ (the first four relations in (9) reproduce also the three-flavour case). The $q$-dependence appears only in the masses of $\omega_8$, $\omega_{15}$, $\omega_{24}$, $\omega_{35}$. Since (isodoublet) particles and their antiparticles must have equal masses, $\gamma_3 = \delta_3$, $\gamma_4 = \delta_4$ in (9), and likewise for $n = 5, 6$. The resulting $q$-MSRs [3] are

$$[n]_{(q)}\ m_{\omega_{n^2-1}} + (b_{n;q} + 2n - 4)\ m_\rho = 2\ m_{D_n^*} + (c_{n;q} + 2) \sum_{r=3}^{n-1} m_{D_r^*} \tag{10}$$

where the notation $[n]_q/[n-1]_q \equiv [n]_{(q)}$ is used and

$$b_{n;q} \equiv n\ c_{n;q} - 6\ [n]_{(q)}^2 + \left(\frac{24}{[2]_q} - 1\right)[n]_{(q)} \qquad c_{n;q} \equiv 2\ [n]_{(q)}^2 - \frac{8}{[2]_q}[n]_{(q)}.$$

This set of $q$-deformed MSRs contains relations (1) at $q \to 1$, as it should. The $q$-analogues show that the coefficients with masses are obtained from their 'classical' prototypes in a more complex way than simply by replacing $a \to [a]_q$.

At $n = 3$ equation (10) yields the $q$-analogue of GMO relation:

$$m_{\omega_8} + \left(2\frac{[2]_q}{[3]_q} - 1\right)m_\rho = 2\frac{[2]_q}{[3]_q} m_{K^*}. \tag{11}$$

Here an important difference is apparent between the case of $n = 3$ and MSRs (10) with more flavours: the $q$-GMO relation depends on $q$ through *the ratio* $[3]_{(q)}$ *only*, while higher MSRs ($n \geqslant 4$) contain *both the ratio* $[n]_{(q)}$ *and the quantity* $[2]_q$. This difference is caused by the presence of non-simple-root elements in the mass operator (8) in all cases except $n = 3$. As mentioned above, definitions of non-simple-root elements and rules of their commutation with simple-root ones are controlled by $q$-Serre relations (4).

If $[3]_q = [2]_q$, the $q$-deformed mass formula (11) simplifies and yields

$$m_{\omega_8} + m_\rho = 2m_{K^*}. \tag{12}$$

Setting $m_{\omega_8} \equiv m_\phi$, one recognizes in equation (12) the nonet mass formula of Okubo [8]. This relation agrees perfectly with the data (up to errors of experiment and of averaging over isoplets). What are higher analogues of Okubo's relation? We put $[n]_q = [n-1]_q$, $n = 4, 5, 6$, in equation (10) and obtain them:

$$m_{\omega_{15}} + (5 - 8/[2]_{q_4})m_\rho = 2\ m_{D^*} + (4 - 8/[2]_{q_4})m_{K^*} \tag{13}$$

$$m_{\omega_{24}} + (9 - 16/[2]_{q_5})m_\rho = 2\ m_{D_b^*} + (4 - 8/[2]_{q_5})(m_{D^*} + m_{K^*}) \tag{14}$$

$$m_{\omega_{35}} + (13 - 24/[2]_{q_6})m_\rho = 2\ m_{D_t^*} + (4 - 8/[2]_{q_6})(m_{D_b^*} + m_{D^*} + m_{K^*}). \tag{15}$$

Here $q_n$ denote the values that solve equations $[n]_q - [n-1]_q = 0$, namely,

$$q_n = \mathrm{e}^{\mathrm{i}\pi k/(2n-1)} \qquad k = \pm 1, \pm 2, \ldots. \tag{16}$$

Note that the quantities $[n]_q - [n-1]_q$, being the polynomials $P_n(q)$ that satisfy the conditions [9] (i) $P_n(q) = P_n(q^{-1})$, (ii) $P_n(1) = 1$, coincide (only formally?) with the Alexander polynomials $\Delta(q)\{(2n-1)_1\}$ of toroidal $(2n-1)_1$-knots. Namely, $[2]_q - 1 = q + q^{-1} - 1 \equiv \Delta(q)\{3_1\}$ corresponds to the trefoil (or $3_1$-) knot, $[3]_q - [2]_q = q^2 + q^{-2} - q - q^{-1} + 1 \equiv \Delta(q)\{5_1\}$ corresponds to the $5_1$-knot, and so on. Due to this, all the $q$-dependence in masses of $\omega_{n^2-1}$ and in coefficients of MSRs (10) can be expressed in terms of various Alexander polynomials:

$$\frac{[3]_q}{[2]_q} = 1 + \frac{\Delta\{5_1\}}{[2]_q} = 1 + \frac{\Delta\{5_1\}}{\Delta\{3_1\}+1}$$
$$\frac{[4]_q}{[3]_q} = 1 + \frac{\Delta\{7_1\}}{[3]_q} = 1 + \frac{\Delta\{7_1\}}{\Delta\{5_1\}+\Delta\{3_1\}+1}$$

etc. The values of (16) may be viewed as roots of respective Alexander polynomials.

The difference between (12) ($n = 3$) and (13)–(15) is manifest. At fixed $n \geqslant 4$, values of additional $q$-number $[2]_{q_n} = 2\cos\frac{k\pi}{2n-1}$, present in (13)–(15), obviously differ for different $|k|$ in (16). However, specific data for masses [10] put into MSR (13) or (14) (with $J/\psi$ and $\Upsilon$, respectively, in place of $\omega_{15}$ and $\omega_{24}$) can satisfy the MSR only with one value of $q_n$. For example, relation (14) *holds* (perfectly, up to errors of data and averaging over isoplets) just *with the primitive* 18th *root of unity taken for* $q_5$, that is, with $k = 1$. Thus, the $q$-deuce in MSRs (10) which originates from $q$-Serre relations (4), serves to 'select' a unique appropriate value of the deformation parameter from set (16). In the case of $U_q(u_3)$, $q$-Serre relations are out of play, so the (extra) $q$-deuce is absent in equation (11) and all the values $q_3 = e^{i\pi k/5}$, $k = \pm 1, \pm 2, \ldots$ are appropriate.

To summarize, we have demonstrated with a concrete example of application of a quantum counterpart of higher-rank Lie algebra, that $q$-Serre relations are important to fix the (unique) physically appropriate value of deformation parameter. Remark also that, although we use (the $(4n-2)$th) roots of unity (16) for $q$, the specific representations exploited within this approach remain irreducible.

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